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Reentrant Dimensional Crossover in Planar Ising Superlattices

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The $(d=2)$-dimensional spin model composed of alternating strips of two different Ising magnets is revisited. Application of modern techniques results in explicit exact solution for the free energy and correlation lengths of the continuous (field-theoretic) limit of the model. In agreement with earlier results, the specific heat generally exhibits *three* different critical features: two near the respective critical temperatures T_{c1} and T_{c2} of the composing models, as well as a new superlattice transition at $T_{c1} < T_c < T_{c2}$. Further analysis shows that at all temperatures between T_{c1} and T_{c2} the correlations in the system include a low-amplitude but very long-range component reflecting fluctuations in a large-scale domain wall network. The essential features of the solution can be explained by a reentrant dimensional crossover from the $d=2$, bulk behavior within the strips, to one-dimensional criticality in the individual strips, and finally back to the two-dimensional behavior on a new, superlattice level. This qualitative understanding of the physical content of the model allows for semiquantitative description of the temperature dependence of spontaneous magnetization and magnetic susceptibility, which have been previously obscure.

KEY WORDS: Critical phenomena; superlattices; magnetic multilayers; dimensional crossover; exact solutions; layered Ising models.

1. INTRODUCTION

The solution given by Onsager to the problem of a spatially homogeneous planar Ising model[1] inspired numerous variations on the main theme. Many problems obtained by including various spatial inhomogeneities into the original model turned out to be to a certain degree solvable as well.[2] Those solutions as well as their subsequent scaling analysis provided a major contribution to the current understanding of the effects of finite size,[2–4] surfaces,[2, 5] and disorder.[2, 6]

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One of the models allowing for an explicit evaluation of its free energy is a planar Ising *superlattice*. It can be generally defined as an Ising lattice in which the nearest-neighbor spin couplings vary *periodically* with *one* of the two spatial coordinates. Its simplest realization is a lattice composed of alternating strips of two different homogeneous Ising ferromagnets. An explicit, albeit rather complicated expression for its free energy was first obtained by Fisher and Ferdinand[7] and then analyzed in more detail by Hahn and coworkers.[8] However, the physical origin of the rather unusual phenomena found to take place in the model has remainded to a large degree obscured.

Our interest in the model is twofold. On one hand, recent progress in creating artificial magnetic heterostructures[9] requires an adequate development of the theory. While the related experimental and theoretical effort has been so far mostly concentrated on the properties of the resulting low-temperature phases, one may expect that the increasingly precise experiments will soon raise the question of universal features associated with the *critical phenomena* in magnetic multilayers. To this end the exact results for the planar Ising superlattice may provide a valuable insight into the new physics characteristic of superlattices. Besides qualitative understanding of the general principles which may apply to three-dimensional systems as well, the results of this paper are *directly* relevant to experiments on thin magnetic films cut out of a $d=3$ superlattice, or deposited on a periodically modulated substrate.

On the other hand, the problem of phase transitions in multilayers is of considerable fundamental importance as well. The periodic reduced temperature field $t(z)$ appearing in a magnetic superlattice due to the difference in the strengths of spin couplings of the composing magnets arranged in layers perpendicular to the "growth direction" z provides a particularly simple and yet nontrivial example of a critical state perturbed by a strong and rapidly varying *relevant* field. While, as has been already mentioned above, particular examples of this situation, i.e., models with surfaces, disorder, etc., have been thoroughly studied before, the current understanding of the response to a *general* form of such perturbation is still largely incomplete. This has been underscored recently by the failure of the standard models of disorder to account for phenomena observed in porous materials.[10]

Based on the standard scaling ideas one may anticipate that the qualitative behavior at a phase transition in a superlattice is determined by the relative value of two lengths: the period of the superlattice l and the correlation length $\xi \propto \Delta t^{-\nu}$ determined by the *amplitude* Δt of the oscillating part of t. Here ν is the correlation length exponent of the bulk components, as we consider the simplest case of two alternating layers belonging to the same bulk universality class. We will see, however, below for the specific case of the $d=2$ Ising universality class that, at least in the *thick* layer limit, $l \gg \Delta t^{-\nu}$, the resulting behavior is quite nontrivial and is not determined by the bulk scaling alone.

The general problem of describing the response of a critical system to inhomogeneous relevant fields has been formulated in refs. 11–13, where a natural variational approach to the problem has been proposed. Indeed an effective local variational principle has been found for the $d=1$ Ising model[14, 11] and for the class of $d=2$ Ising models in zero magnetic field with arbitrary variation of the bond strengths, $t=t(z)$, along a certain z direction.[12, 13] The latter result has been based on an earlier work[15] demonstrating that the partition sum of this class of models factorizes into a product over a set of fictitious $d=1$ Ising chains labeled by the wavenumber q along the other, y direction, in which the system remains homogeneous.

In what follows, these new techniques are applied to the *continuous*, field-theoretic limit of the planar Ising superlattice, as defined in refs. 11 and 13 for the general class of *layered* Ising models. We will obtain simple exact explicit expressions for the free energy of the system, as well as for the energy (i.e., domain wall) density $\varepsilon(z)$. More importantly, studying the evolution of the energy density *components* $\varepsilon_q(z)$ in the long-wavelength limit $q \to 0$, one obtains important information about the behavior of the system on different length scales. On its basis we arrive at the qualitative picture of *reentrant*, $(d=2) \to (d=1) \to (d=2)$-dimensional crossover. It allows us to explain the main features of the exact solution as well as to predict the scaling behavior of the spontaneous magnetization and magnetic susceptibility, the exact evaluation of which is presumably a very hard task.

It has to be stressed that the aim of this paper is to elucidate the universal long-wave length properties of a planar superlattice arising from the interplay between the critical fluctuations in its components. The results presented here by no means exclude much richer structures which may arise when the width of the strips become comparable to the atomic scale (or, say, a period of antiferromagnetic ordering[9]) and the strong microscopic interactions become involved. Correspondingly, this consideration is restricted to the limit $\Lambda l_1, \Lambda l_2 \gg 1$, where Λ is a microscopic wavelength defining the cutoff in our formulation.[13] The basic question arising here is whether despite the fact that the strips are "thick" on this microscopic scale one can still observe effects beyond a simple linear superposition of the intensive properties of the two components. In what follows we give a rather complete long-wavelength description of a superlattice in terms of the physical quantities which can be measured on the individual *bulk* components.[2] The continuum formalism developed in refs. 11 and 13 seems to be ideally suited for that. As discussed in ref. 13, while taking the continuum limit certainly entails losing information about phenomena at the microscopic level, the gain is universality: one expects the results obtained here to be valid for the long-wavelength properties of any periodically modulated system from the planar Ising universality class. An extra benefit[13] is the equivalence of this limit to the many-body quantum mechanical problem of a spin-1/2 chain in a transverse field which slowly varies either in *time* or in space.

2. EXACT RESULTS FOR THE CONTINUUM MODEL

2.1. Definition of the Model

We consider a planar Ising model composed of alternating strips of two components labeled with $j = 1, 2$. The widths of the strips are l_1 and $l_2 = l - l_1$. In terms of the formalism of refs. 12 and 13 the problem is defined by the values t_1 and t_2 which the scaled temperature field t takes within the corresponding strips. The function $t(z)$ therefore has the form of a periodic telegraph signal: $t(z) = t_1$ for $nl < z < nl + l_1$ and $t = t_2$ for $nl + l_1 < z < (n+1)l$, for all integers n. The parameters t_j have the dimension of inverse length and are normalized in such a way that at any given temperature T their absolute values are equal to inverse correlation lengths ξ_j of the corresponding bulk (homogeneous) components[12,13]:

$$t_j(T) = \pm 1/\xi_j(T) \tag{1}$$

where the plus and minus signs correspond to $T > T_{cj}$ and $T < T_{cj}$, respectively. In terms of the original nearest-neighbor Ising model on a rectangular lattice,[12,13] the "temperatures" t_j are directly related to the microscopic spin coupling strengths $K_j^{\perp} = J_j^{\perp}/k_B T$ and $K_j^{\parallel} = J_j^{\parallel}/k_B T$ via $t_j = -[\ln \tanh(K_j^{\perp}) + 2K_j^{\parallel}]/a_{\perp}$, where the superscripts $\parallel$, $\perp$ correspond to the bonds oriented parallel and perpendicular to the *layers*, i.e., the y axis, and $a_{\perp}$ is the microscopic lattice spacing along z. However, the relation (1) between the t_j and the corresponding bulk correlation lengths allows for direct identification of the t-field of a *real* superlattice constructed of components belonging to the planar Ising universality class. If the bulk correlations of the components have been measured, they form a direct input for the theory without any need to identify the bare, microscopic couplings.

[2] Note that "bulk" in this paper actually means "measured in a homogeneous film" of the given component.

To complete the definition of the model we should point out that the interfacial bonds connecting the neighboring strips will be generally different from those of the two components, which in our formalism[13] leads to additional δ-function contributions to the t-field which are located at the interfaces. Such δ-functions, however, can be viewed as an additional, third type of layer in the superlattice, of thickness $l_3 \to 0$. In what follows we will concentrate on the simplest, two-component problem, which, as one will be able to see below, contains all the essential physics necessary for understanding the multicomponent generalizations as well. Note also that the δ-pieces are expected to be absent[13] in the case of diffuse interfaces, i.e., of the interactions varying smoothly (on atomic scale) across the interface.

Finally we note that due to the intrinsic anisotropy of the superlattice geometry it is natural[13] to define the original lattice model on a rectangular lattice allowing for anisotropic spin couplings, $K_j^{\perp} \neq K_j^{\parallel}$, in both components. On the phenomenological level that implies anisotropic bulk correlation lengths, $\xi_j^{\perp} \neq \xi_j^{\parallel}$. However, as shown in ref. 13, we may consider the simplest case of isotropic components, $\xi_j^{\perp} = \xi_j^{\parallel} \equiv \xi_j(T) \equiv |t_j|^{-1}$ without any loss of generality. The anisotropic case is immediately restored by the following mnemonic rule justified in[13]: substitute $|t_j| = 1/\xi_j^{\parallel}(T)$ everywhere except for the cases where t_j enters multiplied by l_j; the absolute values of the latter have to be redefined via $|t_j l_j| \equiv l_j/\xi_j^{\perp}(T)$. We will see below, however, that putting the strips of even perfectly isotropic components together in a superlattice results in anisotropic correlation lengths, $\xi^{\perp} \neq \xi^{\parallel}$, at the scales larger than the period l of the superlattice; this anisotropy is intrinsic to the problem and cannot be removed by a simple rescaling.

2.2. Free Energy and Transverse Correlation Length

To proceed one has to recapitulate some basic results of refs. 15 and 11–13. Rewriting the partition function of the problem as a trace over the product of transfer matrices along the z direction and making transformation to fermionic operators, one can simultaneously reduce[15] *all* z-dependent transfer matrices to a diagonal-block form. Each 2×2 block describes the subspace of fermions with the given absolute value of the wave vector q along the y direction. In the continuum long-wavelength limit introduced in refs. 11 and 13, these symmetric matrices generate a continuous unitary transformation according to

$$\mathbf{T}(q, z) - \mathbf{1} = [t(z)\sigma_3 + q\sigma_1]\, dz \tag{2}$$

where $\sigma_{1,3}$ are the standard Pauli matrices. This establishes equivalence to a set of noninteracting one-dimensional Ising chains (in their respective continuous limits: ref. 11) for which $t(z)$ and q play the roles of the (position-dependent) magnetic field and fugacity of domain walls, respectively. We will call these one-dimensional models "q-chains" below.

Within each of the strips, $j=1, 2$, the transformation generated by the $SU(2)$ matrices (2) can be viewed as a rotation by an *imaginary* angle proportional to the width of the strip. Therefore the cumulative transfer matrix $\mathbf{T}_j(q)$ resulting after multiplication of $\mathbf{T}(q, z)$ within the jth strip has two eigenvalues $\exp[\pm l_j \kappa_j(q)]$, with $\kappa_j(q)=(t_j^2+q^2)^{1/2}$. The rotation axis lies in the (1, 3) plane in the space of the Pauli matrices making angle $\phi_j(q)=\tan^{-1}(t/q)$ with the 1-axis.

It now follows straightforwardly from the Pauli matrix algebra that the combined action of these two rotations applied periodically in the superlattice is always again a rotation by an imaginary angle proportional to the length of the system: $...\mathbf{T}_1\mathbf{T}_2\mathbf{T}_1\mathbf{T}_2... = ...\mathbf{T}_0\mathbf{T}_0...$, where the basic matrix $\mathbf{T}_0(q)=\mathbf{T}_1(q)\,\mathbf{T}_2(q)$ has eigenvalues of the form $\exp[\pm l\kappa(q)]$. The "angular velocity" $\kappa(q)$ is determined via a simple generalization of the stereometric "cosine theorem":

$$\cosh l\kappa = \cosh l_1\kappa_1 \cosh l_2\kappa_2 + \cos(\phi_1-\phi_2)\sinh l_1\kappa_1 \sinh l_2\kappa_2 \tag{3}$$

where all quantities are q-dependent. Following the basic ideas of the transfer matrix formalism,[11] one establishes that $\kappa(q)$ gives the free energy per unit length of the q-chain by $F(q)= -k_B T\kappa(q)$, so that the free energy per unit area of the superlattice is

$$F= -k_B T\int_q \kappa(q) \tag{4}$$

At the same time $\kappa(q)$ is the inverse of the correlation length $\xi(q)$ of the q-chain, so that a simple one-dimensional hyperscaling relation $F(q)= -k_B T/\xi(q)$ is satisfied by each chain. Naturally, the correlation length $\xi_\perp$ of the system along the z direction (i.e., *perpendicular* to the strips) is determined by the largest $\xi(q)$ over all q. As $\kappa(q)$ defined by (1) is always minimal at $q=0$, the inverse correlation length $\xi_\perp^{-1}$ of the superlattice in the direction across the strips is given by

$$\xi_\perp^{-1}(T)=|l_1t_1+l_2t_2|/l=|l_1\xi_1^{-1}(T)\pm l_2\xi_2^{-1}(T)|/l \tag{5}$$

where [cf. (1)] the plus and minus signs in the second expression on the right-hand side correspond to the two components being on the same ($T<T_{c1}<T_{c2}$ or $T_{c1}<T_{c2}<T$) or on different ($T_{c1}<T<T_{c2}$) sides of

their bulk critical points, respectively. The advantage of this second expression in (5) is that it represents the answer via directly measurable correlation lengths $\xi_1(T)$, $\xi_2(T)$ of the *bulk* components (see footnote 2).

2.3. Energy Density Components

The variational formalism of refs. 11–13 has not been really used so far: the principal result (3) has been obtained by combination of the mapping onto the set of one-dimensional q-chains[15] with the continuous limit of those.[11, 13] The variational principle yields, however, much more information: it allows calculation of the q-components $\varepsilon_q(z)$ of the energy density profile $\varepsilon(z)$. The computation is straightforward: within each strip the field t is uniform, so that the general solutions of the Euler–Lagrange equations which equilibrium profiles $\varepsilon_q(z)$ have to satisfy are readily available.[12, 13] Taking into account that the middle point z_{nj} of any strip of the jth component is a center of symmetry of the system, one obtains within that strip

$$\varepsilon_q(z) = \cos(\theta_j)\sin(\phi_j) + \sin(\theta_j)\cos(\phi_j)\cosh[2\kappa_j(z - z_{nj})] \qquad (6)$$

where all quantities are q-dependent, j characterizes the component of which the given strip is made, and the only two remaining unknowns are the parameters θ_j defined by matching the solutions (8) at the interfaces between the strips. Matching leads to a biquadratic equation yielding

$$\sin[\theta_1(q)] = \frac{\sin(\phi_2 - \phi_1)/\sinh(\kappa_1 l_1)}{[c_1^2 + c_2^2 + 2\cos(\phi_2 - \phi_1)c_1 c_2 - \sin^2(\phi_2 - \phi_1)]^{1/2}} \qquad (7)$$

and $\cos(\theta_1) > 0$. Here $c_j(q) = 1/\tanh(\kappa_j l_j)$, $j = 1, 2$, and the profile within the second type of strip results after the obvious permutation of indices. Note that the permutation of indices. Note that the permutation changes the sign of the relative angle $\phi_2 - \phi_1$, so that if the profile in one type of strip is convex, than that in the other is concave. A straightforward integration of $\varepsilon_q(z)$ over q gives then the full energy profile $\varepsilon(z)$. Integrating $\varepsilon_q(z)$ over z, one obtains the full *energy* of the q-chain, which is in turn related to the free energy (3), (4) by a single integration over the temperature. While this is a far more difficult way of arriving at the result (3), (4) than the derivation given in the previous subsection, we will see below that the profiles $\varepsilon_q(z)$ contain plenty of additional information about the behavior of the system.

3. ANALYSIS: CRITICAL TEMPERATURE, SPECIFIC HEAT, AND CORRELATION LENGTHS

From (5) one immediately finds, in agreement with the previous results,[7,8] that the correlation length diverges and the superlattice is critical when the total temperature field per unit length, $t_0 \equiv (l_1 t_1 + l_2 t_2)/l \equiv 1/\xi_\perp(T)$, vanishes. This condition of criticality can be again rewritten via (1), (5) in a simple universal form [note that $t_1 = 1/\xi_1(T)$, but $t_2 = -1/\xi_2(T)$ near T_c]

$$l_1/\xi_1(T_c) = l_2/\xi_2(T_c) \equiv g_c \tag{8}$$

which allows for prediction of the critical temperature of the superlattice when the temperature dependences of the correlation lengths in the bulk components are known.

Equation (8) defines a dimensionless scaling parameter $g_c \sim l\,\Delta t^\nu$, where (cf. Introduction) $\Delta t = (t_1 - t_2)/2 \sim 1/\xi_1(T_{c2}) \sim 1/\xi_2(T_{c1})$ is the amplitude of the variation of the temperature field. The correlation length exponent is $\nu = 1$ in the planar Ising universality class. This parameter enters scaling functions characterizing the criticality in the system qualitatively separating the *thin-strip*, $g_c \ll 1$, and the *thick-strip*, $g_c \gg 1$, regimes.

A particularly important quantity is the specific heat per unit area, which follows from the standard thermodynamic relation $C = -T\,d^2F(T)/dT^2$. In the general case the temperature dependence of the free energy $F(T)$ is implicitly defined by (3), (4), resulting in $F = F(t_1, t_2)$, and by empirical temperature dependences of the bulk correlation lengths, to which the t_j are directly related via (1). In the important limit of the two components having critical temperatures close to each other, $T_{c1} - T_{c2} \ll T_{c1,2}$, one can use $t_{1,2} = t_0 \pm (2l_{2,1}/l)\,\Delta t$, where Δt is approximately constant and t_0 varies linearly with the external temperature throughout the whole range $T_{c1} < T < T_{c2}$. Note that despite the condition $T_{c1} - T_{c2} \ll T_{c1,2}$, the parameter g_c still can be large if the strips are sufficiently thick. With this simplification one can explicitly evaluate the specific heat (up to a non-universal metrical factor):

$$C \sim \int_0^\Lambda \frac{\partial^2 \kappa(t_0, \Delta t, l_1, l_2)}{\partial t_0^2} \frac{dq}{2\pi} \tag{9}$$

In the limit $\Lambda \to \infty$ the function

$$\mathscr{C}(t_0 l, g_c, l_1/l_2) = \lim_{\Lambda \to \infty} [C(t_0, \Delta t, l_1, l_2, \Lambda) - \ln \Lambda/2\pi] \tag{10}$$

becomes universal, representing the singular part of the specific heat for the whole universality class of planar Ising superlattices. The singular part of the specific heat is shown in Fig. 1 against the reduced temperature t_0 in the case of a symmetric superlattice, $l_1 = l_2 = l/2$ (the asymmetry of the superlattice does not seem to have important consequences for its qualitative behavior). Different curves correspond to different periods of

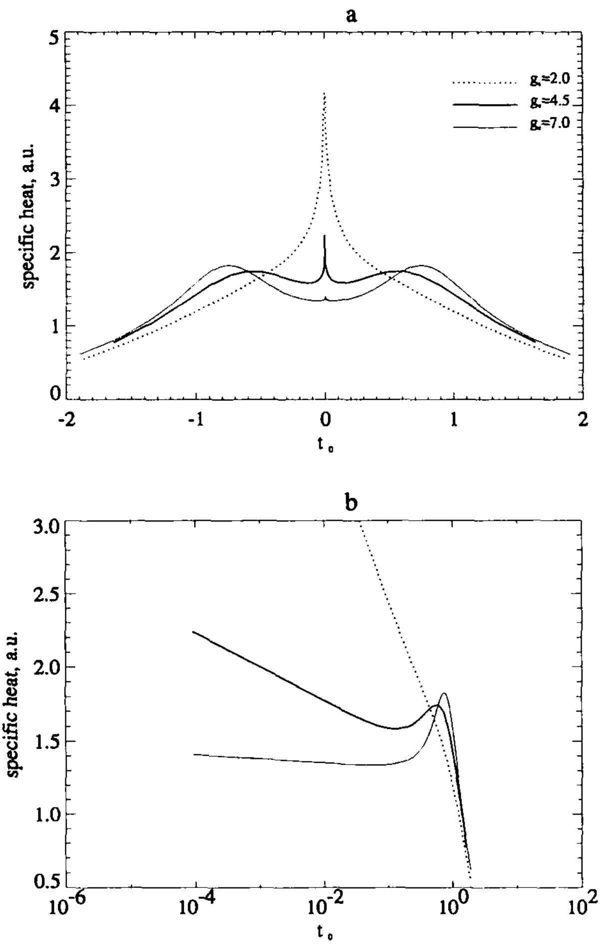

Fig. 1. Critical part of the specific heat vs. temperature curves for a sequence of symmetric, $l_1 = l_2 = l/2$, superlattices in (a) normal and (b) semilogarithmic coordinates. The superlattices are composed of the same pair of components: the reduced critical temperature difference is $\Delta t = 1$, while the thickness of the strips varies as measured by the scaling parameter g_c. The log-normal plot in (b) shows the logarithmic divergence of the central peak.

the superlattice l at fixed $\Delta t = 1$. This corresponds to an experimentally relevant situation in which the pair of components is fixed, while the period of the superlattice may be varied by preparation of the sample. One can clearly see that the variation in $g_c \approx \Delta t l$ leads to evolution in the form of the temperature dependence of the specific heat from one broad central peak in the thin-film limit of small g_c to a *triple*-peak structure for thick strips/large g_c. The two side peaks in the latter case are rounded, while the central one is truly divergent. As one can also see from the logarithmic plots (Fig. 1b), the divergence is logarithmic for any g_c, but its amplitude rapidly falls as g_c increases. This triple-peak structure was obtained earlier by Hahn and coworkers[8] on the basis of their approximate treatment of the thick-strip limit.

Both thick- and thin-strip limits can be described analytically. In the case of a thin-strip superlattice for all $q, t \ll l^{-1}$ Eq. (3) reduces to

$$\kappa^2(q) = t_0^2 + q^2 = \xi_\perp^{-2} + q^2 \tag{11}$$

with $\xi_\perp$ defined in (5). This is nothing but the bulk law (cf. expressions given above for $\kappa_j, j = 1, 2$) with the weighted average (5) standing in place of the scaled temperature. Physically this means that as soon as the correlation lengths of the components grow to the order of the period of the superlattice, the fluctuations wipe out the layered structure, reducing the superlattice to an effectively homogeneous model.

The situation in the thick-strip limit is quite different and in fact more interesting. Near the critical temperature, say T_{c2}, of the second component, which has *higher* ordering temperature and thus stronger ferromagnetic couplings, the correlation length of the other component, ξ_1, is too short to effectively couple the near-critical strips to each other. Correspondingly the rounded specific heat peak appearing at T_{c2} (Fig. 1) asymptotically (as $g_c \to \infty$) reduces to that of a *free* film of the second component. A free film, being an important object in its own right, can be described within our model by taking the limit[12,13] $t_1 \to \infty$ in which the bonds connecting consequent 2-strips are completely broken. The thermodynamic properties of the free film are then determined from (3): $F_0 = -k_B T \int_q \kappa_0(q)$, where

$$l_2 \kappa_0 = \ln[\cosh(l_2 \kappa_2) + \sin(\phi_2) \sinh(l_2 \kappa_2)] \tag{12}$$

Here the (divergent) contribution from the 1-strips has been subtracted to obtain the proper free-film limit. The resulting specific heat function,

$$C_0(t) = \partial^2 F_0 / \partial t_2^2 = \mathscr{C}_0(l_2/\xi_{2z}(T)) + \ln(\Lambda)/(2\pi) \tag{13}$$

with universal dimensionless $\mathscr{C}_0$, grows logarithmically until $\xi_2(T)$ exceeds the order of the film thickness l_2. At this point C_0 reaches the value

$$C_0 \approx C_0(T_{c2}) \approx \ln(l_2 \Lambda)/(2\pi) \tag{14}$$

and rounds up. Due to the symmetry of the system under the change of the sign of the temperature field, $t \to -t$ (which can be traced back to the self-duality of the planar Ising model), the specific heat peak at T_{c1} in the thick-strip limit is described by the same scaling function C_0, although in this latter case the system is equivalent to a sequence of noninteracting films with *frozen* rather than free boundaries. In a free film the maximum of the specific heat is shifted along the $t_2 l_2 = \pm l_2/\xi_2$ axis toward low temperatures by a universal amount.[4] Taking into account the presence of the first component (i.e., finite correlation length $\xi_1 = |t_1|^{-1}$) yields a negative correction to the shift, which is, however, exponentially small in $t_1 l_1 \sim g_c$.[8]

In the vicinity of the critical temperature T_c defined via the first equality in (8), the divergent, central peak of the specific heat can be analyzed on the basis of the low-q expansion

$$\kappa^2(q) = \xi_\perp^{-2} + X^2 q^2 + O(q^4) = \xi_\perp^{-2}[1 + (\xi_\parallel q)^2] \tag{15}$$

valid for $q \ll \Delta t$ and $l_1 t_1 + l_2 t_2 \ll 1$. The right-hand side defines the singular part of the correlation length $\xi_\parallel$ of the superlattice measured parallel to the strips. The ratio $X = \xi_\parallel/\xi_\perp$, so that $\xi_\parallel = X t_0^{-\nu} = X t_0^{-1}$, provides a measure for the macroscopic anisotropy of the superlattice. From standard hyperscaling arguments it follows now that the amplitude A of the singular part of the specific heat, $C \approx A \ln(T - T_c)$, is determined by X as well: $A \sim X^{-1}$. Expanding both sides of (3) to the leading order in q^2 near T_c, one obtains

$$X^2 = 1 + \frac{(|t_1|^{-1} + |t_2|^{-1})^2}{l^2}[g_c^2 + \sinh^2(g_c)] \tag{16}$$

This expression describes the crossover exhibited by the superlattice transition at T_c from essentially isotropic behavior, $X \approx 1$, in the thin-strip limit, $g_c \ll 1$, to exponentially strong anisotropy, $X \sim \exp(g_c)/(2g_c)$, in the other extreme, $g_c \gg 1$. Thus in the thick-strip limit the critical component of the diffuse scattering peak from the superlattice is predicted to be extremely narrow in the direction parallel to the strips, implying very long-range correlations in that direction. On the other hand, the amplitude of the central specific heat peak, as evident from Fig. 1, rapidly decreases in the same limit,[8] while the side peaks grow. The central peak represents, however, the only true divergence in the system for any finite g_c.

4. ANALYSIS: ENERGY DENSITY COMPONENTS

Additional insight into the problem is obtained by observing the evolution of the partial energy profiles $\varepsilon_q(z)$ given by (6), (7), with decreasing wave number $q \to 0$ (Fig. 2). As discussed elsewhere,[16] this evolution appears to exhibit many features of a *functional* renormalization group flow.[18] It always starts from a homogeneous *ultraviolet stable* (i.e., stable

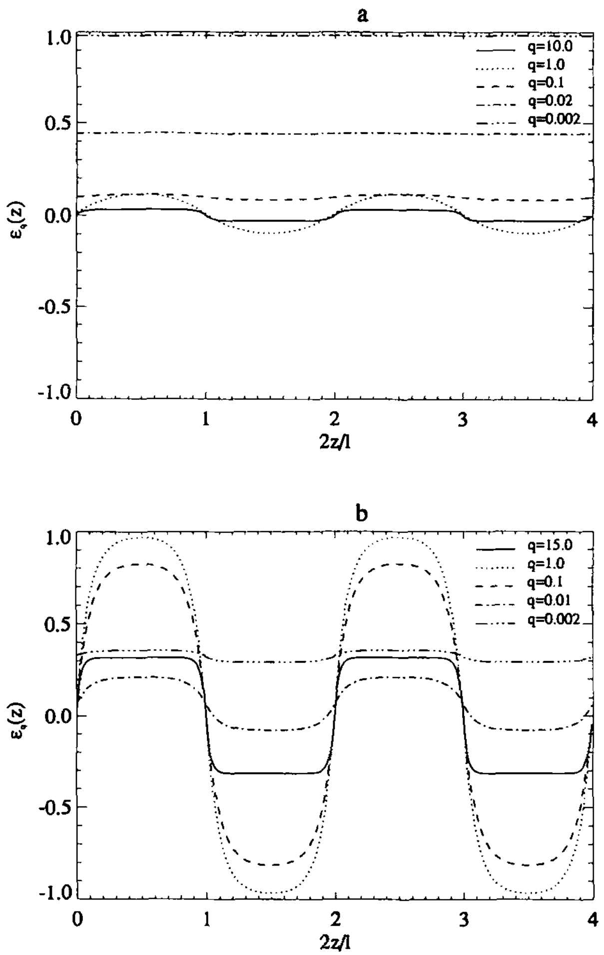

Fig. 2. Evolution of the partial energy density profile $\varepsilon_q(z)$ with decreasing wavenumber q for the value of the overall reduced temperature $t_0 = 0.1$ in (a) thin-strip, $g_c = 0.3$, and (b) thick-strip, $g_c = 5.0$, superlattice. The coordinate z and the wavenumbers q in both plots are scaled by the thickness of a single layer $l_1 = l_2 = l/2$.

as $q \to \infty$) fixed-point profile $\varepsilon_q = 0$. This fixed point is unstable in the other, long-wavelength limit, where, except for the critical point, the profiles approach one of the *infrared stable* homogeneous fixed points: $\varepsilon_q = +1$ and $\varepsilon_q = -1$, describing the high- and the low-temperature states of the system, respectively. In the case of a thin-strip superlattice, $g_c \ll 1$, the critical domain is essentially defined by $|t_0| < l^{-1}$. It encompasses both bulk critical temperatures and the whole interesting region between them. Within this critical range the evolution of $\varepsilon_q(z)$ proceeds essentially uniformly in the whole system (Fig. 2a) and is completed within the range of scales $\Lambda > q > \xi_{\parallel} = \xi_{\perp}$, the latter being the only important length scale in the problem.

The situation is again more interesting, however, in the thick-strip limit at temperatures $T_{c1} \leqslant T \leqslant T_{c2}$ (Fig. 2b). As q decreases below a value of order $\Delta t \sim \xi_j^{-1}$, the energy density components come close to the two different fixed point values, $\varepsilon_q = +1$ and $\varepsilon_q = -1$, within the strips of the first and the second components, respectively. These quasihomogeneous regions are separated by the kinklike solutions describing an interface between the high- and the low-temperature phases.[12, 13] The characteristic interface width is of order $\Delta t^{-1} \ll l_1, l_2$ in the thick-strip limit under consideration. This inhomogeneous structure remains almost unchanged over an exponentially wide range of scales $X^{-1} \Delta t < q < \Delta t$. If the system is not particularly close to the criticality at T_c, at still larger scales the evolution proceeds according to the "winner takes all" scenario: if $|t_1 l_1| < |t_2 l_2|$, then the $\varepsilon_q = +1$ domains flip over to the dominant $\varepsilon_q = -1$ state dictated by the second component or vice versa. If, however, $\xi_{\perp} \gg l$, i.e., the system is *close* to criticality, then at $q \gtrsim X \Delta t$ the profiles first get back close to the critical fixed point $\varepsilon_q \equiv 0$ and only at still larger scales, $q \gtrsim \xi_{\parallel}^{-1} \equiv (X\xi_{\perp})^{-1}$, the system evolves to one of the massive, $\varepsilon_q = \pm 1$, states. Thus within the temperature range of width $\sim l^{-1}$ around T_c the system is characterized by three characteristic length scales along the strip direction describing three different crossovers: (1) formation of the $\varepsilon_q = \pm 1$ domains at the scale $\xi_b \sim \Delta t^{-1}$ of the order of the bulk correlation lengths of the components, $\xi_b \approx \xi_j(T_c)$, (2) smearing out of these domains at much larger crossover scale $\xi_c \approx X\xi_b$, and, finally, (3) attraction by one of the noncritical states at $\xi_{\parallel} = X\xi_{\perp}$ happening uniformly throughout the superlattice. The latter scale is the true correlation length diverging at the phase transition at T_c [cf. (5)]. Note that the second crossover scale, ξ_c, depends essentially on the properties of a single strip, but not on the global balance between the values $|t_1 l_1|$ and $|t_2 l_2|$ measuring the tendencies to formation of the high- and the low-temperature phases. That balance defines the scales $\xi_{\perp, \parallel}$ characterizing the third crossover, as well as the ultimate destination of the flow.

The nature of the second crossover and of the exponential dependence of length ξ_c on the parameter $g_c \sim l/\xi_b$ is most clearly seen from the analysis of the free-film limit, $t_1 = +\infty$, $\xi_1 = 0$. In this case ε_q takes the high-temperature value $+1$ everywhere outside the $j=2$ film for all q. At $t_2 > 0$, i.e., $T > T_{c2}$, the ε_q profile inside the film monotonically increases from zero at large q to the background value, $\varepsilon_q \equiv +1$, at small q. However, below the bulk critical temperature, $T < T_{c2}$, at $q \lesssim |t_2|$ the profile of $\varepsilon_q(z)$ develops a domain of the low-temperature state, $\varepsilon_q \equiv -1$, inside the film. The latter holds until q decreases below the value $\xi_c^{-1} \approx |t_2| \exp(l_2 |t_2|)$ at which the domain flips over to the high-temperature phase value $+1$ enforced on it by the environment. At this point we recall the relation[3] $\xi_2(T) = k_B T/2\Sigma_2(T)$ connecting the correlation length in the low-temperature phase of the planar Ising model to the linear tension of a domain wall separating domains of the opposite *spin* orientations.[4] This leads to the expression

$$\xi_c \approx \exp[2\Sigma_2(T) l_2/k_B T] \tag{17}$$

The exponential form on the right-hand side of (20) has been shown[(19)] to give the leading, exponential factor in the temperature variation of correlation length of an Ising film (i.e., strip) at low temperatures. The underlying physics is that at large scales the strip becomes equivalent to a *one*-dimensional Ising model. Indeed, the inverse length $1/\xi_c$ is nothing but the Boltzmann factor (fugacity) of a pair of domain walls across the film. The latter is known (e.g., ref. 20) to be the elementary excitation leading to disordering of the linear Ising model at low temperatures. Note that formally, in the language of the $\varepsilon_q(z)$ profiles, the crossover happens when the value $\sim \exp(-t_2 l_2)$ which the exponential tale of the interface (see footnote 4) adds to the profile in the bulk of the film, becomes of order of the difference $|-1-\varepsilon_{q2}|$ between the bulk energy density value $\varepsilon_{q2} = t_2/(t_2^2+q^2)^{1/2}$ and the completely ordered state $\varepsilon_q = -1$. Recalling that ε_q essentially represents the density of excitations (i.e., domain wall loops) of size q^{-1}, we can interpret ξ_c as the scale at which the fluctuations induced by the boundaries of the film start dominating the intrinsic, bulk ones.

[3] The relation follows from the simple "bubble" picture of correlations, see, e.g., ref. 18, Section 11: the dominant contribution to the correlation between two spins at a distance r apart comes from the configuration in which both spins are included in a single domain having the form of a narrow finger of length r. The two walls of the finger contribute the energy $2\Sigma_2 r$ to the Boltzmann factor $\exp(-2\Sigma_2 r/k_B T) = \exp(-r/\xi_2)$.

[4] It is important to keep in mind that the domain walls between different spin orientations in the low-temperature phase of a uniform system, with which we are dealing at the moment, are completely distinct from the interfaces between the high- and the low-temperature phases in different strips appearing due to inhomogeneities in the scaled temperature field t.

Returning to the original problem of a thick strip superlattice, we arrive at the following physical picture (Fig. 3). As the temperature is decreased toward the higher of the bulk critical temperatures of the two components, T_{c2}, the superlattice behaves as a simple mixture of the bulk components until the correlation length of the stronger, second component grows to the value of the order of the strip width l_2. After this happens the spins of the second component are essentially correlated across each strip, and the strips become equivalent to one-dimensional Ising chains. An individual strip (film) remains, however, disordered at any positive temperature due to creation of domain walls across the strip, which break the strip into a sequence of domains of opposite spin orientations. At scales smaller/larger than the typical domain size ξ_c the strip is essentially

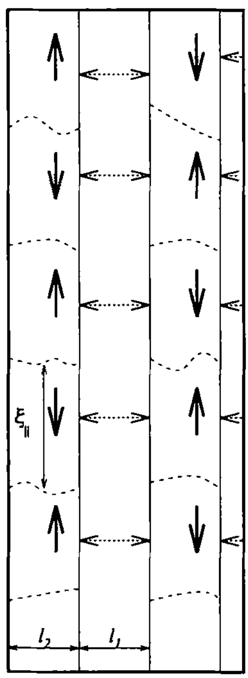

Fig. 3. Schematic picture of spin–spin correlations in a thick-strip superlattice in the temperature range $T_c < T < T_{c2}$. Strips of the more strongly coupled, second component are broken into long domains with different spin orientations. The dotted lines symbolize exponentially weak coupling between domains in neighboring strips via fluctuations in the first component. Near T_c the value of this coupling *per domain* becomes $O(1)$, leading to formation of superstructural domains of the third level, and eventually to a long-range order in the superlattice.

ordered/disordered, which explains the values $\varepsilon = \pm 1$ the energy components take within the regions occupied by the second component at q larger/smaller than ξ_c^{-1}. In a superlattice, however, the domains of the second component are coupled across the strips of the weaker, first component due to the finite value of the correlation length $\xi_1 = t_1^{-1}$ of the latter. The value of this coupling per unit spin is proportional to $\exp(-l_1/\xi_1)$, which is the value of a perturbation at one boundary of the $j = 1$ strip carried to the other boundary by the correlations in the $j = 1$ component. It is very small away from T_{c1}. However, the real fluctuating unit at $T_{c1} < T < T_{c2}$ is not a single spin, but a correlated domain of length ξ_c. The relevant energy scale then is the coupling per single *domain*, $\propto \xi_c \exp(-l_1/\xi_1) \propto \exp(l_2/\xi_2 - l_1/\xi_1)$. The critical temperature T_c is thus [exactly! cf. (8)] determined as the one at which the exponentially large domain size ξ_c compensates the exponential weakness of couplings through the paramagnetic strips. The latter then suffice to promote coherence between domains in different strips and thus in the whole system. The exponential dependence of the critical amplitude on the parameter $g_c \equiv l_j/\xi_j$ appears to be a direct consequence of the $\xi \propto \exp(2E/k_B T)$ correlation length growth law characteristic of the one-dimensional Ising model, with the energy of an effective domain wall of the latter, $E = \Sigma_2 l_2$, being proportional to the width of the strip. The picture is completed by applying the duality, $t \to -t$, to characterize the behavior at the low-temperature side of the transition. The nontrivial physical content of the problem comes from the reentrant dimensional crossover: the three regimes in the evolution of the profiles of ε_q described above are characteristic of two-dimensional, one-dimensional, and then again two-dimensional physics.

5. SPONTANEOUS MAGNETIZATION AND MAGNETIC SUSCEPTIBILITY

We now show how our qualitative understanding of the problem allows for some useful estimates going beyond the information provided by the exact solution. In particular, computation of the spontaneous magnetization in inhomogeneous planar Ising models is a very hard task; exact results when available come in so complicated a form that extracting useful information out of them constitutes a hard problem of its own. However, the simple scaling arguments given above can be developed into the following semiquantitative picture for a thick-strip system.

As our superlattice exhibits the only true phase transition at $T = T_c$, the spontaneous magnetization averaged over the whole system is exactly zero at any $T > T_c$. However, before the emergence of true long-range order, within the temperature range $T_c < T \lesssim T_{c2}$ a hierarchy of large domains of the same spin orientation is formed in the system (Fig. 3). As the temperature is decreased toward T_{c2} the events start with appearance of critical spin clusters within the strips of the second component. In a given cluster, spins have a single preferred orientation, but due to the fractal structure of the cluster the absolute value of magnetization density (per unit area) decreases with the cluster size according to the renormalization group equation

$$dm/d\ln\xi = -\omega_m m \tag{18}$$

Here ξ is the size of the cluster, or more generally, the running length scale of our renormalization group flow, and ω_m is the scaling dimension of magnetization, $\omega_m^{d=2} = 1/8$ in the planar Ising universality class. The equation has to be solved with the initial condition $m(\Lambda^{-1}) = m_0$, where m_0 is the microscopic magnetization density of the order of one spin per unit cell. Above T_{c2} the size of these basic critical clusters is given by the bulk correlation length $\xi_2(T) = t_2^{-1}$ which grows with decreasing temperature. While the clusters are disoriented at scales exceeding l_2, a weak external magnetic field h aligns each of them with probability $hm(\xi_2)\xi_2^d/k_B T$, thus inducing global magnetization density $m = h_\chi \approx (l_2/l)\, hm^2(\xi_2)\xi_2^d/k_B T$. The factor l_2/l accounts for the fact that the important magnetization fluctuations are localized within the strips of the second component. This standard argument[21] gives the correct susceptibility growth law $\chi = dm/dh = \chi_0(\Lambda\xi_2)^{d-2\omega_m} \propto t_2^\gamma$, with $\gamma = (d-2\omega_m)/\omega_t = (\omega_h - \omega_m)\nu = 7/4$ in the Ising universality class. Here the scaling dimension of temperature, $\omega_t = 1/\nu = 1$, and the standard relation, $\omega_h + \omega_m = d = 2$, between the scaling dimensions of the (here magnetization) density and the conjugate (here, correspondingly, magnetic) field have been employed.

Below T_{c2}, in a bulk sample (i.e., homogeneous film) of the second component the clusters are *correlated* at scales larger than ξ_2 (which now *decreases* with decreasing temperature). Thus the value of magnetization saturates at

$$m_2(T) = m_0[\Lambda\xi_2(T)]^{-\omega_m} \approx M_0\,|t_2|^\beta \tag{19}$$

Here the spontaneous magnetization exponent is $\beta = \omega_m/\omega_t = 1/8$ and $M_0 \approx m_0\Lambda^{-\omega_m}$ is the critical amplitude of magnetization in the second component. In the spirit of this paper, rather than operating with the bare microscopic parameter m_0, we will often use below the empirical magnetization curve of the second component, $m_2(T)$, as the initial condition for the renormalization group flow equation (18) at scales greater than $\xi_2(T)$.

The phenomena developing in a superlattice are more complicated. With the temperature decreasing toward T_{c2}, the size of the clusters increases until, at $t_2 \sim 1/l_2$, it reaches the limits set by the boundaries of a strip. At this point, i.e., at $T_{c2}^+ = T_{c2}[1 + O(l^{-1})]$, each strip is broken into roughly equilateral domains of size l_2 with the absolute value of magnetization density $\approx m_0(\Lambda l_2)^{-\omega_m}$ in each of them. The corresponding value of susceptibility is $\chi \approx \chi_0(\Lambda l_2)^\gamma$, $\gamma = 7/4$. This state of the system is not significantly changed until the temperature is lowered to T_{c2}^- defined via $t_2 \approx -1/l_2$. At that point the first bifurcation in the temperature dependence of the spin–spin correlations takes place. As the temperature keeps decreasing toward T_c the size of the basic critical clusters in the second component starts decreasing proportional to the bulk correlation length $\xi_2(T)$ providing increasing magnetization density value $m_2(T)$ at scales larger than $\xi_2(T)$. At the same time, an emerging system of domain walls breaks the strips of the second component into long and narrow domains of essentially $d = 1$ Ising nature (Fig. 3). The length $\xi_{||}$ of these new domains starts growing according to Eq. (17), as the energy of the domain walls $\Sigma_2(T)$ increases in the low-temperature phase of the second component. These new clusters are essentially critical clusters of the linear Ising model; therefore to determine the value of magnetization in them one would need to integrate Eq. (18) with $\omega_m = \omega_m^{d=1}$ characteristic of the $d = 1$ Ising behavior. However, as $\omega_m^{d=1} = 0$,[11] these $d = 1$ Ising clusters are *not* fractal and $m_2(T)$ characterizes the magnetization density at all scales between ξ_2 and $\xi_{||}$. Note that as these second-level clusters, or more precisely correlations in the system, are anisotropic, one has to use $\xi_{||} = 1/q$, the scale in the direction along the strip, in which the system is uniform, as the running variable of the renormalization group equations, just as has been already done in the previous section. To complete the description one can in fact complement (18) with a somewhat trivial equation

$$d\xi_\perp / d \ln \xi_{||} = \omega_\perp \xi_\perp \tag{20}$$

where $\omega_\perp = 1$ and 0 in the two- and one-dimensional regimes, respectively. On scales larger than $\xi_{||}$ above T_c different clusters are uncorrelated, so that the spontaneous magnetization is absent, while the susceptibility is estimated as above:

$$\chi \approx (l_2/l)\, m_2(T)[m_2(T)\, l_2 \xi_{||}(T)/k_B T] \tag{21}$$

The susceptibility keeps growing below T_{c2} mostly due to the exponential increase in $\xi_{||}$, (17), with the growth of $m_2(T)$, (19), adding an additional growing power-law prefactor.

This one-dimensional behavior spans the wide temperature range $T_{c2}^{-} > T > T_c^{+}$ with $T_c^{+} = T_c[1 + O(l^{-1})]$. The latter is defined by the condition that the correlation length $\xi_\perp = t^{-1}$ exceeds the period of the superlattice l. Below T_c^{+} the second-level clusters in neighboring strips start interacting strongly, forming $d = 2$ critical clusters characteristic again of the planar Ising behavior. Near T_c the length of the second-level clusters reaches the value ξ_c and the absolute value of magnetization density is $m \approx m_2(T_c) \approx m_0(\Delta t/\Lambda)^\beta$; both do not change significantly within the critical interval $t \lesssim l^{-1}$. The two correlation lengths $\xi_{\|}$, $\xi_\perp$ of the system, which diverge at T_c, are now associated with the two dimensions of the new, third-level clusters. According to (20), where $\omega_\perp = 1$ again, both lengths grow similarly at this stage, $\xi_{\|} \propto \xi_\perp = t^{-1}$. However the difference in the initial conditions, $\xi_\perp(\xi_{\|} = \xi_c) \approx l \ll \xi_c \sim e^{g_c}$, appearing due to inhibited growth of $\xi_\perp$ in the second, quasi-one-dimensional regime, results in the constant anisotropy factor $\xi_{\|}/\xi_\perp = \xi_c/l = X$ in full agreement with the exact results above. The absolute value of magnetization density within a third-level cluster follows from (18), (19) as

$$m \approx m_2(T_c)(l/\xi_\perp)^{\omega_m} = m_2(T_c)\,|tl|^\beta = m_2(T_c)\,|l_1/\xi_1(T) - l_2/\xi_2(T)|^\beta$$
$$\sim m_0\, g_c^\beta\, |t|^\beta \tag{22}$$

Above T_c in this regime the susceptibility is

$$\chi \approx m^2 \xi_{\|} \xi_\perp / k_{\mathrm{B}} T \approx m_2^2(T_c)(tl)^{2\beta}\, X t^{-2}/k_{\mathrm{B}} T \sim X(g_c)\, g_c^{2\beta} \chi_0 t^\gamma \tag{23}$$

One can see that at large g_c, the critical amplitude of susceptibility is much larger than that of the bulk second component. Below T_c, the third-level clusters are ordered, so that $m(T)$ is nothing but the spontaneous magnetization of the superlattice. Again the critical amplitude is increased with respect to the original amplitude M_0 of the second component by a large factor g_c^β. As t decreases to the value $\sim -1/l$, i.e., at $T_c^{-} = T_c[1 - O(l^{-1})]$, the steeply rising magnetization curve of the superlattice reaches the value of order $m_2(T)$. At that point the third-level clusters shrink to the size of the second-level clusters. That means that the latter get ordered, and from the point of view of the magnetization fluctuations, both third- and second-level clusters disappear, leaving essentially bulk-like structure in the strips of the second component. Near T_{c1} the total magnetization of the superlattice gets an additional contribution from ordering in the strips of the first component. The latter can be viewed as a phase transition in external field induced by the second component, which can be analyzed similarly to the above.

6. DISCUSSION

In summary, an explicit exact solution for the free energy and energy density components of the continuum limit of a planar Ising superlattice in zero magnetic field has been obtained and analyzed. The analysis leads to a rather complete long-wavelength description of the planar Ising superlattices; it includes a prediction for the critical temperature of the superlattice, and for the temperature dependences of the specific heat and of the (anisotropic) correlation lengths. All results take universal form after being expressed through the correlation lengths $\xi_j(T)$ of the two *bulk* components ($j = 1, 2$), which are supposed to be measured independently. Where applicable, our conclusions agree with those of earlier papers.[7,8] Two predictions for the *thick-strip* limit of the model have to be emphasized: one is the existence of *three* separate peaks in the temperature dependence of the specific heat (Fig. 1), another is the presence of exponentially large correlation length in a wide temperature interval between the bulk critical temperatures T_{c1} and T_{c2} of the components.

Our analysis leads further to the qualitative picture of reentrant, $d = 2 \rightarrow d = 1 \rightarrow d = 2$-dimensional crossover underlying these unusual properties of the model. The essential content of the crossover is that fluctuations on different scales (i.e., with different wavelengths) exhibit properties characteristic of different spatial dimensionalities. The one-dimensional character of fluctuations in the intermediate regime leads further to the extreme anisotropy of the reentrant, two-dimensional fluctuations at the largest scales. At different temperatures different fluctuations dominate the behavior of the system. One may say that the two side peaks of the specific heat curve (Fig. 1) indicate creation of the domain structure up to the scale of the period of the superlattice. The large, exponentially growing correlation length accompanied with the *decrease* in the value of the specific heat then reflects essentially one-dimensional fluctuations in this large-scale structure. Finally, as the correlations break through *across* the layers, the critical two-dimensional ordering is displayed by the central divergent specific heat peak. The utility of this qualitative understanding of the problem is further demonstrated by calculating the crossovers and critical amplitudes for the spontaneous magnetization and magnetic susceptibility in the problem. One should note here anomalously large critical amplitudes of both.

Besides direct relevance to potential experiments on magnetic *films* (cut out of, or deposited upon, a slice of a three-dimensional superlattice), it is argued in a separate publication[22] that the qualitative picture of reentrant dimensional crossover holds and allows for a rather complete scaling picture of critical phenomena in real three-dimensional superlattices.

In particular, very similar effects of crossover between $(d-1)$- and d-dimensional behavior are to be expected for a $d=3$ superlattice composed of two magnets belonging to the Heisenberg universality class, as the latter also lacks ordered phase while exhibiting exponentially large correlation length in $d=2$. The functional renormalization group flow realized by evolution of the partial energy density profiles, $\varepsilon_q(z)$, which we used above to clarify the physical contents of the problem, appears to be useful far beyond the specific superlattice geometry. In fact, in another paper[16] we demonstrate that it allows for a rather complete classification of critical behavior in a general type of planar layered Ising model (as defined in ref. 13), including the celebrated McCoy–Wu random multilayer. One must, however, exercise certain care in using the planar superlattice as a model for three-dimensional systems: it is well know that the domain wall excitations playing the central role in our approach here are not sufficient in describing the onset of long-range order in higher dimensions.

Four further comments are in order.

1. The qualitative large-scale behavior of the superlattice in the temperature range $T_c \lesssim T \lesssim T_{c2}$ is similar to that characteristic of highly anisotropic *elementary* rectangular lattices which have been studied previously (ref. 1; ref. 21, Section IV.9, and references cited there). Such a lattice can be obtained by an obvious integrating out of the fluctuations on scales smaller than l, resulting in the strength of the weak bonds of order $e^{-l_1/\xi_1} \sim e^{-g_c} \gg 1$. This can be visualized by shrinking the domains appearing in Fig. 3 to elementary cites of a new lattice. Note that all the nontrivial properties of the superlattice are expressed via the anisotropy parameter X. This comes at no surprise of course: while one could hardly expect that a periodic modulation of the temperature field in the problem could become a *relevant* perturbation changing the universality class of the transition, the anisotropy is known to be a *marginal* perturbation with respect to the planar Ising renormalization-group fixed point.

2. While the analysis of Section 5 operates with the intuitively appealing picture of nested critical (fractal) clusters, real quantitative description has to be formulated in terms of the spin–spin correlation functions. Different types of clusters then correspond to different components of the correlation function dominating correlations over different distance ranges. Those components are likely to be in one-to-one correspondence with singularities exhibited by the Fourier transform of the spin correlation function.[23] The imaginary part of such a singularity defines the inverse corelation lengths of the corresponding component of the correlation function. Near T_c we expect to find at least four different singularities: relatively high up on the imaginary axis are the two branch points corresponding to

the noncritical bulk correlations in the two components, further down is the one responsible for extremely anisotropic ($d = 1$)-like correlations along the strips, and finally, the lowest branch point yields the most long-range correlations sweeping across the superlattice. The crossover temperatures T_2^-, T_c^+, etc., may then be formally defined via bifurcations in the correlation length spectrum.

3. It is clear that as the short-range correlations dominate the fluctuations at smaller scales, their *amplitudes* have to be much larger than those of the long-range components, reflecting that only a small fraction of the degrees of freedom of the system are involved in the latter. This is clearly seen also from the size of the different peaks of the specific heat (Fig. 1) if one remembers the relation between the specific heat and the energy–energy correlation function: the long-range, critical component of these correlations leading to the divergent specific heat peak at T_c has an amplitude which is exponentially small in the thick-strip limit.

Note that in the same limit an exponentially large correlation length persists over the temperature range of order unity, thus defying the usual necessity of fine tuning to the critical regime. The price one has to pay, though, is the exponentially small amplitude of that "generically critical" component. Note that different *response* functions, i.e., the specific heat and the magnetic susceptibility, determined by the zero-wavenumber component of the corresponding correlation functions, may be either small, like the specific heat, or large, like the susceptibility, depending on the relative values of the correlation lengths and amplitudes.

4. Finally, we have in fact achieved our goal of expressing the properties of a superlattice composed of thick strips via macroobservables of the components. However, unlike the simple additive thermodynamics of Gaussian systems, the critical state of matter considered here is characterized by an essentially *nonlinear* dependence of the thermodynamic quantities of the composite structure on those of the individual components.

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